Generalized formula for the first derivative of the electric-field intensity

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This paper presents a derivation for a general formula relating the first derivative of the electric-field intensity in any direction to the first derivative of the direction cosines in the orthogonal direction. A restricted form of the formula valid only in the direction of the field lines was first stated by Thomson, and later proved by others. The general form proved here promises to be useful in other attempts to apply techniques of differential geometry to the rapid solution of field problems.

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INTRODUCTION

Since 1985, a few attempts have appeared making use of concepts from differential geometry to calculate variables in the electrostatic field [1–4]. Estevez and Bhuiyan [1] presented a power-series solution to a problem given by Jackson [5] and originally stated by Thomson [8]. The problem requires the proof that at the surface of a charged conductor, the normal derivative of the magnitude of the electric field E satisfies

$$\frac{\partial \|E\|}{\partial n} = -\|E\| \left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{1}$$

where R_1 and R_2 are the principal radii of curvature of the surface at the point under consideration, and n is the normal direction to the surface at the same point. Later, Pappas [2] presented an elegant proof for the same formula. Recently, Zhou [4] used this formula to obtain relations for field magnitudes at two points along a flux line in the electrostatic field.

In this paper, it will be shown that formula (1) is, in fact, a special case of a general formula that gives the derivative of the electric-field intensity in *any* direction within the field, not necessarily the direction normal to an equipotential surface.

The need for a generalized formula for the first derivative of the field intensity arose from attempts to solve Laplace's equation geometrically. While such an application is beyond the scope of this paper, it is to be pointed out that $\partial\|E\|/\partial n = |\partial^2 V/\partial n^2|$, where V is the potential. When Laplace's equation is considered, we further note that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

at any point; however, $\partial^2 V/\partial n^2 \neq 0$ at the same point, where n is the direction of the flux line. Therefore, it was necessary to find a general formulation for the second derivative of potential when the coordinate system is arbitrarily oriented, i.e., a formulation for the components $\partial^2 V/\partial x^2$, etc. of Laplace's equation.

A GENERALIZED FORMULA FOR THE FIRST DERIVATIVE OF THE ELECTRIC-FIELD INTENSITY

Although the following derivation can be carried in any number of dimensions, a two-dimensional (2D) analysis will be given here to avoid the mathematical complexity which obscures the underlying physics. The corresponding 3D analysis is summarized in the Appendix. The derivation assumes a charge-free region inside which Laplace's equation holds.

Consider an infinitesimal segment ds of an electric field line S, as in Fig. 1 (see Ref. [6]). ds carries an electric field of intensity E, having components E^x and E^y along the principal directions. We have

$$E^x = \|E\| \frac{dx}{ds} = \|E\| \, \delta^x,$$
$$E^y = \|E\| \frac{dy}{ds} = \|E\| \, \delta^y, \tag{2}$$

where δ^x, δ^y denote the components of a unit vector tangent to the field line, and $\|E\| = \sqrt{(E^x)^2 + (E^y)^2}$. Taking partial derivatives of the field components in (2) in the principal directions gives

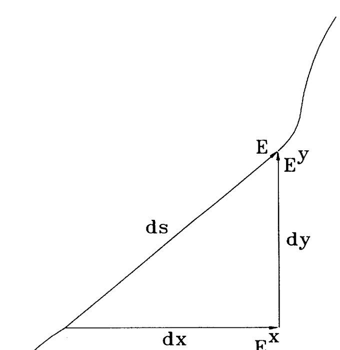

FIG. 1. An infinitesimal segment ds of a field line S is resolved into its components dx and dy. ds carries a field of intensity E.

$$\frac{\partial E^x}{\partial x} = \|E\| \, \frac{\partial \delta^x}{\partial x} + \delta^x \, \frac{\partial \|E\|}{\partial x},$$
$$\frac{\partial E^y}{\partial y} = \|E\| \, \frac{\partial \delta^y}{\partial y} + \delta^y \, \frac{\partial \|E\|}{\partial y}. \tag{3}$$

The field intensity $\|E\|$ has partial derivatives given by

$$\frac{\partial \|E\|}{\partial x} = \frac{1}{2\|E\|}\left[2E^x \frac{\partial E^x}{\partial x} + 2E^y \frac{\partial E^y}{\partial x}\right] = \delta^x \, \frac{\partial E^x}{\partial x} + \delta^y \, \frac{\partial E^y}{\partial x},$$
$$\frac{\partial \|E\|}{\partial y} = \delta^x \, \frac{\partial E^x}{\partial y} + \delta^y \, \frac{\partial E^y}{\partial y}. \tag{4}$$

From Eqs. (3) and (4),

$$\frac{\partial E^x}{\partial x} = \|E\| \, \frac{\partial \delta^x}{\partial x} + (\delta^x)^2 \, \frac{\partial E^x}{\partial x} + \delta^x \delta^y \, \frac{\partial E^y}{\partial x},$$
$$\frac{\partial E^y}{\partial y} = \|E\| \, \frac{\partial \delta^y}{\partial y} + (\delta^y)^2 \, \frac{\partial E^y}{\partial y} + \delta^x \delta^y \, \frac{\partial E^x}{\partial y},$$

or

$$\frac{\partial E^x}{\partial x} \, (\delta^y)^2 = \|E\| \, \frac{\partial \delta^x}{\partial x} + \delta^x \delta^y \, \frac{\partial E^y}{\partial x},$$
$$\frac{\partial E^y}{\partial y} \, (\delta^x)^2 = \|E\| \, \frac{\partial \delta^y}{\partial y} + \delta^x \delta^y \, \frac{\partial E^x}{\partial y}, \tag{5}$$

where the identity

$$(\delta^x)^2 + (\delta^y)^2 = 1 \tag{6}$$

(the sum of the squares of the direction cosines) has been used.

To evaluate $\partial E^x/\partial y$ and $\partial E^y/\partial x$, we note that the analytic solution V of Laplace's equation in rectangular coordinates is separable and can be expressed as a sum of products [7]

$$V = X_1 Y_1 + X_2 Y_2 + \cdots,$$

where X_n and Y_n are functions of x and y, respectively. Thus we have

$$-E^x = \frac{\partial V}{\partial x} = \frac{dX_1}{dx} Y_1 + \frac{dX_2}{dx} Y_2 + \cdots,$$
$$-E^y = \frac{\partial V}{\partial y} = \frac{dY_1}{dy} X_1 + \frac{dY_2}{dy} X_2 + \cdots,$$

so that

$$\frac{\partial E^x}{\partial y} = \frac{\partial E^y}{\partial x} = -\left(\frac{dX_1}{dx}\frac{dY_1}{dy} + \frac{dX_2}{dx}\frac{dY_2}{dy} + \cdots\right). \tag{7}$$

From Eqs. (5) and (7),

$$\frac{\partial E^x}{\partial x} \, (\delta^y)^2 - \frac{\partial E^y}{\partial y} \, (\delta^x)^2 = \|E\| \, \left[\frac{\partial \delta^x}{\partial x} - \frac{\partial \delta^y}{\partial y}\right]. \tag{8}$$

Now, we note that Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

can be written as

$$\frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} = 0. \tag{9}$$

Hence, from Eqs. (9), (8), and (6)

$$\frac{\partial E^y}{\partial y} = \|E\| \, \left[\frac{\partial \delta^y}{\partial y} - \frac{\partial \delta^x}{\partial x}\right]. \tag{10}$$

From Eqs. (10) and (3), we see that

$$\delta^y \, \frac{\partial \|E\|}{\partial y} = - \, \|E\| \, \frac{\partial \delta^x}{\partial x}$$

or

$$\frac{\partial \|E\|}{\partial y} = - \, \|E\| \, \frac{\partial \delta^x/\partial x}{\sqrt{1-(\delta^x)^2}}. \tag{11}$$

Similarly,

$$\frac{\partial \|E\|}{\partial x} = - \, \|E\| \, \frac{\partial \delta^y/\partial y}{\sqrt{1-(\delta^y)^2}}.$$

To understand these relations, consider an equipotential line V and a field line E normal to V at a given point. For an arbitrarily oriented coordinate system X-Y (as in Fig. 2, where the direction cosines δ^x, δ^y are given by $\delta^x = \cos\theta = \sin\psi$, $\delta^y = \cos\psi = \sin\theta$), Eq. (11) gives the first partial derivative of the electric-field intensity in one direction (for instance, $\partial\|E\|/\partial y$) as a function only of the first derivative of the direction cosine along the orthogonal direction (in this case, $\partial\delta^x/\partial x$).

To relate (11) and its 3D counterpart to formula (1), observe first that in 2D, formula (1) is given along the Y direction by $\partial\|E\|/\partial y = -\|E\|(1/R)$, where R is the radius of curvature of the equipotential surface in 2D. Now, we observe that

$$\frac{\frac{\partial \delta^x}{\partial x}}{\delta^y} = \frac{\frac{\partial \sin\psi}{\partial x}}{\cos\psi} = \frac{d\psi}{dx}.$$

The last expression is precisely the definition of curvature for a planar curve.

If the coordinate system is selected such that the Y axis is taken along the electric-field direction, then $d\psi/dx = d\psi/ds = 1/R$ defines the curvature of the equipotential surface V, where s is the arc length of the curve. This selection of axes reproduces Eq. (1) in 2D. For other orientations of the coordinate system $d\psi/dx$ is

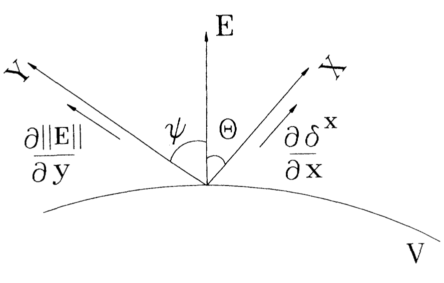

FIG. 2. The field E is normal to the equipotential line V at all points. At the point considered, E is resolved into an arbitrary coordinate system $X-Y$; the derivative of the field intensity in one direction (here Y) is given in terms of the derivative of the direction cosine in the orthogonal direction.

not a measure of curvature; however, Eq. (11) enables the computation of $\partial\|E\|/\partial y$ in such cases.

Comparable expressions can be derived in three dimensions; this is summarized in the Appendix. The resulting formula (A10) relates the partial derivative of the field strength in an arbitrary direction z to derivatives of the direction cosines in the other two ordinate directions. In the special case where field derivative is taken in the field direction (i.e., z aligns with E, and $\delta^z = 1$), then both x and y are normal to E, so the direction cosines δ^x and $\delta^y = 0$. This yields the form

$$\frac{\partial\|E\|}{\partial z} = -\|E\|\left(\frac{\partial\delta^x}{\partial x} + \frac{\partial\delta^y}{\partial y}\right).$$

Equation (1) is recovered by noting that the two quantities $\partial\delta^x/\partial x$, and $\partial\delta^y/\partial y$ represent the reciprocal of the two principal radii of curvature of the equipotential surface.

APPENDIX

In three dimensions, we have

$$\begin{aligned}\frac{\partial E^x}{\partial x} &= \|E\|\frac{\partial\delta^x}{\partial x} + \delta^x\frac{\partial\|E\|}{\partial x},\\ \frac{\partial E^y}{\partial y} &= \|E\|\frac{\partial\delta^y}{\partial y} + \delta^y\frac{\partial\|E\|}{\partial y},\\ \frac{\partial E^z}{\partial z} &= \|E\|\frac{\partial\delta^z}{\partial z} + \delta^z\frac{\partial\|E\|}{\partial z}.\end{aligned} \quad \text{(A1)}$$

By writing Laplace's equation in the form

$$\frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = 0$$

and adding the three independent equations in (A1), we obtain

$$\delta^x\frac{\partial\|E\|}{\partial x} + \delta^y\frac{\partial\|E\|}{\partial y} + \delta^z\frac{\partial\|E\|}{\partial z} = -\|E\|\left(\frac{\partial\delta^x}{\partial x} + \frac{\partial\delta^y}{\partial y} + \frac{\partial\delta^z}{\partial z}\right). \quad \text{(A2)}$$

Now, we rewrite the last equation in (A1) as

$$-\|E\|\frac{\partial\delta^z}{\partial z} = -\frac{\partial E^z}{\partial z} + \delta^z\frac{\partial\|E\|}{\partial z}. \quad \text{(A3)}$$

By using the expression $\|E\| = \sqrt{(E^x)^2 + (E^y)^2 + (E^z)^2}$, we obtain

$$\begin{aligned}\frac{\partial\|E\|}{\partial x} &= \delta^x\frac{\partial E^x}{\partial x} + \delta^y\frac{\partial E^y}{\partial x} + \delta^z\frac{\partial E^z}{\partial x},\\ \frac{\partial\|E\|}{\partial y} &= \delta^x\frac{\partial E^x}{\partial y} + \delta^y\frac{\partial E^y}{\partial y} + \delta^z\frac{\partial E^z}{\partial y},\\ \frac{\partial\|E\|}{\partial z} &= \delta^x\frac{\partial E^x}{\partial z} + \delta^y\frac{\partial E^y}{\partial z} + \delta^z\frac{\partial E^z}{\partial z},\end{aligned} \quad \text{(A4)}$$

from which Eq. (A3) can be written as

$$\begin{aligned}-\|E\|\frac{\partial\delta^z}{\partial z} &= -\frac{\partial E^z}{\partial z} + \delta^x\delta^z\frac{\partial E^x}{\partial z} + \delta^y\delta^z\frac{\partial E^y}{\partial z} + (\delta^z)^2\frac{\partial E^z}{\partial z}\\ &= \left[\frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y}\right]\left[(\delta^x)^2 + (\delta^y)^2\right]\\ &\quad + \delta^x\delta^z\frac{\partial E^z}{\partial x} + \delta^y\delta^z\frac{\partial E^z}{\partial y},\end{aligned} \quad \text{(A5)}$$

where the identity

$$(\delta^x)^2 + (\delta^y)^2 + (\delta^z)^2 = 1$$

and the symmetry relations

$$\frac{\partial E^x}{\partial z} = \frac{\partial E^z}{\partial x}, \qquad \frac{\partial E^y}{\partial z} = \frac{\partial E^z}{\partial y}$$

have been used. By using the first two equations in (A4), Eq. (A5) can be further manipulated to give

$$\begin{aligned}-\|E\|\frac{\partial\delta^z}{\partial z} &= \delta^x\frac{\partial\|E\|}{\partial x} + \delta^y\frac{\partial\|E\|}{\partial y} + (\delta^x)^2\frac{\partial E^y}{\partial y}\\ &\quad + (\delta^y)^2\frac{\partial E^x}{\partial x} - \delta^x\delta^y\left(\frac{\partial E^x}{\partial y} + \frac{\partial E^y}{\partial x}\right).\end{aligned} \quad \text{(A6)}$$

Now, by using the relations

$$\begin{aligned}\frac{\partial E^x}{\partial y} &= \|E\|\frac{\partial\delta^x}{\partial y} + \delta^x\frac{\partial\|E\|}{\partial y},\\ \frac{\partial E^y}{\partial x} &= \|E\|\frac{\partial\delta^y}{\partial x} + \delta^y\frac{\partial\|E\|}{\partial x},\end{aligned}$$

the quantity $\delta^x\delta^y(\partial E^x/\partial y + \partial E^y/\partial x)$ in Eq. (A6) can be written as

$$\begin{aligned}\delta^x\delta^y\left[\frac{\partial E^x}{\partial y} + \frac{\partial E^y}{\partial x}\right] &= 2\delta^x\delta^y\frac{\partial E^x}{\partial y}\\ &= \|E\|\delta^x\delta^y\left[\frac{\partial\delta^x}{\partial y} + \frac{\partial\delta^y}{\partial x}\right]\\ &\quad + (\delta^x)^2\delta^y\frac{\partial\|E\|}{\partial y} + (\delta^y)^2\delta^x\frac{\partial\|E\|}{\partial x},\end{aligned}$$

which, by use of the relations in (A1), can be further expressed as

$$\begin{aligned}2\delta^x\delta^y\frac{\partial E^x}{\partial y} &= \|E\|\delta^x\delta^y\left[\frac{\partial\delta^x}{\partial y} + \frac{\partial\delta^y}{\partial x}\right]\\ &\quad + (\delta^x)^2\frac{\partial E^y}{\partial y} + (\delta^y)^2\frac{\partial E^x}{\partial x}\\ &\quad - \|E\|\left[(\delta^x)^2\frac{\partial\delta^y}{\partial y} + (\delta^y)^2\frac{\partial\delta^x}{\partial x}\right].\end{aligned} \quad \text{(A7)}$$

Substituting from (A7) into (A6), we get

$$\begin{aligned}-\|E\|\frac{\partial\delta^z}{\partial z} &= \delta^x\frac{\partial\|E\|}{\partial x} + \delta^y\frac{\partial\|E\|}{\partial y}\\ &\quad + \|E\|\left[(\delta^x)^2\frac{\partial\delta^y}{\partial y} + (\delta^y)^2\frac{\partial\delta^x}{\partial x}\right.\\ &\quad \left. - \delta^x\delta^y\left(\frac{\partial\delta^x}{\partial y} + \frac{\partial\delta^y}{\partial x}\right)\right].\end{aligned} \quad \text{(A8)}$$

From Eqs. (A2) and (A8), we see that

$$\delta^z \frac{\partial\|E\|}{\partial z} = -\|E\| \left(\frac{\partial\delta^x}{\partial x} + \frac{\partial\delta^y}{\partial y}\right) +\|E\| \left[(\delta^x)^2 \frac{\partial\delta^y}{\partial y} + (\delta^y)^2 \frac{\partial\delta^x}{\partial x} -\delta^x\delta^y \left(\frac{\partial\delta^x}{\partial y} + \frac{\partial\delta^y}{\partial x}\right)\right] \quad \text{(A9)}$$

or

$$\frac{\partial\|E\|}{\partial z} = -\frac{\|E\|}{\delta^z} \left[\frac{\partial\delta^x}{\partial x}[1-(\delta^y)^2] + \frac{\partial\delta^y}{\partial y}[1-(\delta^x)^2] +\delta^x\delta^y \left(\frac{\partial\delta^x}{\partial y} + \frac{\partial\delta^y}{\partial x}\right)\right]. \quad \text{(A10)}$$

This is the desired form of the general equation in three dimensions.

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